More Results on Adjacency Matrix and Energy of a $T_2$ Hypergraph

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ABSTRACT

A hyper graph $H = (X, D)$ is said to be a $T_2$ hyper graph if for any three distinct vertices $u, v$ and $w$ in $X$, there exist a hyper edge containing $u, v$ but not $w$ and another hyper edge containing $w$ but not $u$ and $v$. In this article, the adjacency energy of a $T_2$ hyper graph is studied. It is shown that $AE(H) \leq n\sqrt{n} + n - 4$.

Keywords: $T_2$ hyper graph, adjacency matrix, adjacency energy.

Subject Classification: 05C65

1 Introduction

The basic definitions and terminologies of a hyper graph are not given here and we refer it [1, 2]. The concept of hyper graph was introduced by Berge in 1967. Later the same concept was studied by Vitaly Voloshin and Alain Bretto [2, 3]. Seena and Raji Pilakkat were introduced Hausdorff hyper graph, $T_0$ hyper graph and $T_1$ hyper graph. Based on [4, 5], we introduced a new class of hyper graph namely $T_2$ hyper graph and the parameter adjacency energy is studied for the same. Throughout this article $H$ is a simple connected $T_2$ hyper graph with order $n$ and size $m$. Here the order and size are the minimum number of vertices and edges used to define a $T_2$ hypergraph. In $A(H)$, $\lambda_1$ is the largest eigen value and $\lambda_n$ is the smallest eigen value. The following definitions and theorems are used in sequel.

Definition 1.1. [6] A hypergraph $H = (X, D)$ is said to be a Hausdorff hyper graph if for any two distinct vertices $u, v$ of $X$ there exists hyper edges $D_1$ and $D_2$ such that $u \in D_1, v \in D_2$ and $D_1 \cap D_2 = \emptyset$

Definition 1.2. [4] A hypergraph $H = (X, D)$ is said to be a $T_0$ hyper graph if for any two distinct vertices $u, v$ of $X$ there exists a hyper edge containing one of them but not the other.

Definition 1.3. [5] A hyper graph $H = (X, D)$ is said to be a $T_1$ hyper graph if for any two distinct vertices $u, v$ of $X$ there exists a hyper edge containing $u$ but not $v$ and another hyper edge containing $v$ but not $u$.

Definition 1.4. [7] A hyper graph $H = (X, D)$ is said to be a $T_2$ hyper graph if for any three distinct vertices $u, v$ and $w$ of $X$ there exist a hyper edge containing $u, v$ but not $w$ and another
hyper edge containing w but not u, v.

**Definition 1.5.** [8] The adjacency matrix is the square matrix which rows and columns are indexed by the vertices of H and where for all \( u, v \in X, u \neq v \), \( a_{uv} = |\{d \in D: u, v \in D\}| \) and \( a_{uv} = 0 \).

**Definition 1.6.** [9] The adjacency energy of a hypergraph is sum of the eigen values of its adjacency matrix.

**Definition 1.7.** [9] A graph G on n vertices is said to be hypo energetic if \( E(G) < n \). Graphs for which \( E(G) \geq n \) are said to be non – hypo energetic.

**Definition 1.8.** [9] A graph G on n vertices is said to be hyper energetic if \( E(G) > 2n-2 \).

**Definition 1.9.** [10] A graph G on n vertices is said to be border energetic if \( E(G) = 2n-2 \).

### 2 Adjacency matrix and energy of a T₂ hyper graph

In this section, we find the energy of a \( T_2 \) hyper graph using adjacency matrix. Consider a \( T_2 \) hyper graph given in figure 1 with 10 vertices and 6 edges.

![Figure 1. T₂ Hyper graph](image)

The Adjacency matrix of \( H = T_2 \) is given by

\[
A(H) = \begin{pmatrix}
0 & 2 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 1 \\
2 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 1 \\
1 & 1 & 0 & 2 & 0 & 0 & 2 & 2 & 1 & 1 \\
1 & 1 & 2 & 0 & 1 & 1 & 1 & 1 & 1 & 1 \\
0 & 0 & 0 & 1 & 0 & 2 & 1 & 1 & 1 & 1 \\
0 & 0 & 0 & 1 & 2 & 0 & 1 & 1 & 1 & 1 \\
0 & 0 & 2 & 1 & 1 & 1 & 0 & 3 & 1 & 1 \\
0 & 0 & 2 & 1 & 1 & 1 & 3 & 0 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 2 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 2 & 0
\end{pmatrix}
\]
The eigen values of $A$ (H) are $\lambda = 8.97, 2.9, 1.59, -0.4, -1.29, -2, -2, -2.74, -3$

Therefore, Adjacency energy $AE (H) = \sum_{i=1}^{n} |\lambda_i| = 26.88$

<table>
<thead>
<tr>
<th>Vertices</th>
<th>Energy</th>
<th>$n\sqrt{n} + n - 4$</th>
</tr>
</thead>
<tbody>
<tr>
<td>4</td>
<td>8</td>
<td>8</td>
</tr>
<tr>
<td>5</td>
<td>8</td>
<td>12.18</td>
</tr>
<tr>
<td>6</td>
<td>12</td>
<td>16.69</td>
</tr>
<tr>
<td>7</td>
<td>15.73</td>
<td>21.52</td>
</tr>
<tr>
<td>8</td>
<td>18.84</td>
<td>26.62</td>
</tr>
<tr>
<td>9</td>
<td>25.61</td>
<td>32</td>
</tr>
<tr>
<td>10</td>
<td>26.88</td>
<td>37.62</td>
</tr>
<tr>
<td>11</td>
<td>31.32</td>
<td>43.48</td>
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<tr>
<td>12</td>
<td>40.95</td>
<td>49.56</td>
</tr>
<tr>
<td>13</td>
<td>44.21</td>
<td>55.87</td>
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<tr>
<td>14</td>
<td>143.97</td>
<td>62.38</td>
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<tr>
<td>15</td>
<td>46.38</td>
<td>69.09</td>
</tr>
<tr>
<td>16</td>
<td>61.97</td>
<td>76</td>
</tr>
<tr>
<td>17</td>
<td>69.32</td>
<td>83.09</td>
</tr>
<tr>
<td>18</td>
<td>67.28</td>
<td>90.36</td>
</tr>
<tr>
<td>19</td>
<td>78.91</td>
<td>97.81</td>
</tr>
<tr>
<td>20</td>
<td>100.36</td>
<td>105.44</td>
</tr>
</tbody>
</table>

Table 1. Adjacency energy of a $T_2$ hyper graph

**Result 2.1.** For a $T_2$ hyper graph $AE (H) \leq n\sqrt{n} + n - 4$. Equality holds only if $n=4$ in H

**Proof.** The above table gives the adjacency energy of a $T_2$ hyper graph of order $n$, where $4 \leq n \leq 20$.

**Observation 2.2.** From the above table,

(i) $T_2$ hyper graph is hyper energetic

(ii) $T_2$ hyper graph is border energetic in $n=5$

(iii) $T_2$ hyper graph is non-hyper energetic

**Result 2.3.** Let $H$ be a $T_2$ hypergraph with $4 \leq n \leq 20$ then

$\lambda_1 < \frac{3n}{2} + \sqrt{\frac{3}{2}}$ if $4 \leq n \leq 19$

$[\lambda 1] < \left[\frac{3n}{2} + \sqrt{\frac{3}{2}}\right]$ if $n=20$
Proof: The result follows from the below table

<table>
<thead>
<tr>
<th>vertices</th>
<th>largest eigen value($\lambda_1$)</th>
<th>$\frac{3\mu}{2} + \frac{3}{2}$</th>
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<tr>
<td>4</td>
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<td>7.22</td>
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<tr>
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<td>3.4</td>
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<tr>
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<tr>
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</tr>
<tr>
<td>20</td>
<td>31.28</td>
<td>31.22</td>
</tr>
</tbody>
</table>

Theorem 2.4. Let H be a $T_2$ hypergraph with $4 \leq n \leq 20$ then $AE(H) < \frac{n(n-2)^2}{(det(A(H)))^\frac{1}{n}}$ 

Proof. From an arithmetic and geometric mean inequality,

$$\frac{\sum_{i=1}^{n} |\lambda_i|}{n} > \left( \prod_{i=1}^{n} |\lambda_i| \right)^{\frac{1}{n}}$$

We have $|\lambda_i| > (\prod_{i=1}^{n} |\lambda_i|)^{\frac{1}{n}}$ for all $i=1,2,3,...,n$

Therefore $|\lambda_1| > (\prod_{i=1}^{n} |\lambda_i|)^{\frac{1}{n}}$

$$|\lambda_1| \sum_{i=1}^{n} |\lambda_i| |\lambda_n| \sum_{i=1}^{n} |\lambda_i|$$

$$|\lambda_1| \sum_{i=1}^{n} |\lambda_i| > (\prod_{i=1}^{n} |\lambda_i|)^{\frac{1}{n}} \sum_{i=1}^{n} |\lambda_i|$$

$$|\lambda_1| \sum_{i=1}^{n} |\lambda_i| > n|\lambda_1|^2 > (\prod_{i=1}^{n} |\lambda_i|)^{\frac{1}{n}} \sum_{i=1}^{n} |\lambda_i|$$

$$n(n-2)^2 > (\prod_{i=1}^{n} |\lambda_i|)^{\frac{1}{n}} AE(H)$$

$$AE(H) < \frac{n(n-2)^2}{(det(A(H)))^\frac{1}{n}}$$

Theorem 2.5. Let H be a $T_2$ hypergraph with $4 \leq n \leq 20$ then $A > \frac{n(n-2)}{n-1}$ where $A = \sum_{i=1}^{n} \sum_{j=1}^{m} a_{ij}^2$. 

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Proof. We have $(\sum_{i=1}^{n} |\lambda_i|)^2 \leq (n-1) (\sum_{i=2}^{n} |\lambda_i|^2)$

$$(-\lambda_1^2) \leq (n-1) (\sum_{i=1}^{n} |\lambda_i|^2) - \lambda_1^2$$

$$n\lambda_1^2 \leq (n-1)A$$

$$(n-1)A > n\lambda_1^2 = n(n-2)$$

$$A \geq \frac{n(n-2)}{n-1}$$

**Theorem 2.6.** Let $H$ be a $T_2$ hypergraph with $4 \leq n \leq 20$ then

$$AE(H) < \begin{cases} 
\frac{4n}{(det A)^\frac{1}{n}} & \text{if } n = 4, 5, 6 \\
\frac{9n}{(det A)^\frac{1}{n}} & \text{if } n = 9 \text{ to } 12 \\
\frac{16n}{(det A)^\frac{1}{n}} & \text{if } n = 16 \text{ to } 19 \\
\frac{25n}{(det A)^\frac{1}{n}} & \text{if } n = 20 
\end{cases}$$

**Proof.** From an arithmetic and geometric mean inequality,

$$\frac{\sum_{i=1}^{n} |\lambda_i|}{n} > \left(\prod_{i=1}^{n} \lambda_i\right)^\frac{1}{n}$$

We have $|\lambda_i| > (\prod_{i=1}^{n} \lambda_i)^\frac{1}{n} \forall i = 1, 2, 3, \ldots, n$

Therefore $|\lambda_n| > (\prod_{i=1}^{n} \lambda_i)^\frac{1}{n}$

$$|\lambda_n| \sum_{i=1}^{n} |\lambda_i| > (\prod_{i=1}^{n} \lambda_i)^\frac{1}{n} \sum_{i=1}^{n} |\lambda_i|$$

$$|\lambda_n| \sum_{i=1}^{n} |\lambda_i| = |\lambda_n|( |\lambda_1| + |\lambda_2| + \cdots + |\lambda_n| )$$

$$|\lambda_n| \sum_{i=1}^{n} |\lambda_i| > n\lambda_n^2 > (\prod_{i=1}^{n} \lambda_i)^\frac{1}{n} \sum_{i=1}^{n} |\lambda_i| \quad \cdots \quad (1)$$

Case 1: If $n=4, 5, 6$

Then the smallest eigen value $\lambda_n = 2$

From (1), $4n > (\prod_{i=1}^{n} \lambda_i)^\frac{1}{n} \text{ AE}(H)$

$$\frac{4n}{(det A)^\frac{1}{n}} > \text{ AE}(H)$$

Case 2: If $n=9 \text{ to } 12$

Then the smallest eigen value $\lambda_n = -3$

From (1), $9n > (\prod_{i=1}^{n} \lambda_i)^\frac{1}{n} \text{ AE}(H)$

$$\frac{9n}{(det A)^\frac{1}{n}} > \text{ AE}(H)$$

Case 3: If $n=16 \text{ to } 19$

Then the smallest eigen value $\lambda_n = -4$

From (1), $16n > (\prod_{i=1}^{n} \lambda_i)^\frac{1}{n} \text{ AE}(H)$

$$\frac{16n}{(det A)^\frac{1}{n}} > \text{ AE}(H)$$
Case 4: If $n=20$
Then the smallest eigen value $\lambda_n = -5$

From (1), $25n > (\prod_{i=1}^{n} \lambda_i)^{\frac{1}{n}} AE(H)$

$$\frac{25n}{(detA)^n} > AE(H)$$

**Theorem 2.7.** Let $H$ be a $T_2$ hypergraph with $4 \leq n \leq 20$ then

$$detA(H) > \begin{cases} 
-2^n & \text{if } 4 \leq n \leq 6 \\
-3^n & \text{if } 9 \leq n \leq 12 \\
-4^n & \text{if } 16 \leq n \leq 19 \\
-5^n & \text{if } n = 5 
\end{cases}$$

Proof. In $H$, $\prod_{i=1}^{n} \lambda_i = \lambda_1 \lambda_2 ... \lambda_n \forall i$
Since $\lambda_i > \lambda_n \forall i$
> $\lambda_n \lambda_n ... \lambda_n$
> $\lambda_n^n$ ....................(1)

Case 1: If $4 \leq n \leq 6$ then $\lambda_n = -2$
From (1), $detA(H) > -2^n$
Case 2: If $9 \leq n \leq 12$ then $\lambda_n = -3$
From (1), $detA(H) > -3^n$
Case 3: If $16 \leq n \leq 19$ then $\lambda_n = -4$
From (1), $detA(H) > -4^n$
Case 4: If $n = 20$ then $\lambda_n = -5$
From (1), $detA(H) > -5^n$

**References**

10. Sujitha S., Sharmila D. Adjacency Energy of a $T_2$ Hypergraph. International